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Ginzburg–Landau model for a free-electron laser: from single mode to spikes

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Abstract

Single-mode operation of a free-electron laser is modeled by the Ginzburg–Landau equation. The linear stability of a single-mode solution is analyzed, and connections are established with known instabilities of the Ginzburg–Landau equation. It is found that there is no Benjamin–Feir instability and hence, the principal mode with the largest gain is always stable. However, the Eckhaus (or the phase) instability generally exists for a mode with frequency outside a range centered on the principal mode. This gives rise to two distinct possibilities: either there is spontaneous frequency shifting to the stable mode with the largest growth rate and a consequent tendency to approach single-mode operation, or there is a sudden chaotization and spikiness in the radiation field. Analytical criteria and scaling are given and tested by numerical simulations. © 1998 Elsevier Science B.V. All rights reserved.

The possibility that a free-electron laser (FEL) can produce a powerful and coherent optical beam of a single frequency is potentially of great interest for many applications. There is compelling experimental evidence from the FEL at the University of California at Santa Barbara (UCSB) that such a possibility is realizable with long-pulse electron beams [1]. Although there is some theoretical controversy [2,3] as to whether the FEL at UCSB actually attained a single-mode state, there is unambiguous observational evidence that the optical beam in the experiment exhibited a clear tendency to operate on a very narrow bandwidth. Future experiments using very long electron pulses, such as that at the Center for Research in Electro-Optics and Lasers (CREOL) of the University of Central Florida, are expected to provide definitive evidence of single-mode operation.

In this paper, we report some recent theoretical developments [4] on single-mode operation of an FEL using the Ginzburg–Landau equation (GLE) which has been proposed as a model for the nonlinear evolution of the radiation field [5,6]. The Ginzburg–Landau model was originally motivated by observations of optical "spiking" in several experiments [7–9]. (For a brief review of related theory and experiment in the early 1990s on optical spikes, the reader is referred to Ref. [6]). A qualitative physical mechanism attributing the generation of spikes to the growth of sidebands was outlined by Warren and coworkers [7]. A necessary condition for the excitation of the sideband instability [10] is slippage between the optical and electron beams. However, the experiment at Columbia University [11] did observe spikes (with characteristic width

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similar to that of a solitary-wave solution [5] of the GLE) apparently without the excitation of the sideband instability.

Single-mode operation of the FEL can be realized in experiments with a very long-pulse electron beam when effects involving slippage of the beam can be neglected. It thus represents a good test for the GLE, which has been shown to represent the nonlinear dynamics of the radiation field with reasonable accuracy when the effects of slippage can be neglected [4]. One of the advantages of the GLE is that it has a rich mathematical literature, with antecedents in hydrodynamics [12–14], that can be brought to bear on the issue of stability of single-mode operation. We show by analysis and numerical simulation that the Ginzburg–Landau model provides useful insight on how single-mode operation evolves out of nonlinear interaction of multiple modes. In particular, it yields specific analytical conditions on when single-mode operation is stable, when instability occurs, and the impulsive nonlinear growth of the instability to yield a chaotic and spiky optical beam. We also give quantitative information that may help resolve the controversy as to whether single-mode operation was realised in the UCSB FEL [1].

We refer to Refs. [4,5] for the derivation of the GLE

$$\frac{\partial A}{\partial z} = \mathrm{i}\lambda_0 A - \frac{1}{v_g}\frac{\partial A}{\partial t} - \frac{\mathrm{i}\alpha}{2}\frac{\partial^2 A}{\partial t^2} + \mathrm{i}\beta|A|^2 A. \quad (1)$$

Here A is the dimensionless vector potential of the radiation field, $\lambda_0(\omega_s)$ is the growing (gain $= -\lambda_{0i} > 0$) complex eigenvalue satisfying the usual third-order equation as a function of the reference frequency ω_s,$v_g \equiv c/(1 + c\partial\lambda_0/\partial\omega_s)$, $\alpha \equiv \partial^2\lambda_0/\partial\omega_s^2$, and β is a complex coefficient (with $\beta_i > 0$). We write $\lambda_0 = \lambda_r + \mathrm{i}\lambda_i$, $1/v_g = \mu_r + \mathrm{i}\mu_i$, $\alpha = \alpha_r + \mathrm{i}\alpha_i$ and $\beta = \beta_r + \mathrm{i}\beta_i$, and apply the transformations $A = A_0 A' \exp[\mathrm{i}(Kz' + \Omega t')]$, $z = z_0 z'$, $t = t_0 t' - z/v_0$, where

$$z_0 = 1/(-\lambda_i + \mu_i^2/2\alpha_i), \quad t_0^2 = \alpha_i z_0/2,$$

$$\Omega = \mu_i t_0/\alpha_i,$$

$$1/v_0 = -\mu_r + \alpha_r\mu_i/\alpha_i, \quad K = (\lambda_r - \alpha_r\mu_i^2/2\alpha_i^2)z_0,$$

$$A_0^2 = 1/z_0\beta_i. \quad (2)$$

Eq. (1) then becomes the GLE in standard form

$$\frac{\partial A}{\partial z} = A + (1 + \mathrm{i}c_1)\frac{\partial^2 A}{\partial t^2} - (1 + \mathrm{i}c_2)|A|^2 A, \quad (3)$$

where $c_1 = -\alpha_r/\alpha_i$, $c_2 = -\beta_r/\beta_i$ are real parameters and all primes have been dropped for notational convenience.

Eq. (3) has an exact single-mode solution of the form

$$A = \sqrt{1 - \omega_0^2}\exp\mathrm{i}\{[-c_2 + (c_2 - c_1)\omega_0^2]z - \omega_0 t + \phi_0\}, \quad (4)$$

where ω_0,ϕ_0 are real constants. The frequency of the radiation field is given by $\omega = \omega_s - \mu_i/\alpha_i + \omega_0/t_0$. Note that the maximal gain frequency for the GLE is $\omega_s - \mu_i/\alpha_i$ (with $\omega_0 = 0$), and is a leading-order approximation to the frequency of maximal gain ω_m of the true gain curve, with $\omega_m = \omega_s - \mu_i/\alpha_i + O(|\omega_m - \omega_s|^2)$, if ω_s is close to ω_m itself.

Single-mode solutions of the GLE may suffer two types of instabilities [12–14]. The first type is known as the Benjamin–Feir instability when every mode is unstable. If such an instability were operative in a FEL, it would be impossible to realize single-mode operation. The second type is known as the Eckhaus instability which is said to occur when a single-mode solution is stable if its frequency is within a certain range, but unstable outside of that range. We now analyze these instabilities in the context of FELs.

We first consider the solution with $\omega_0 = 0$. (We show below that if the mode with maximum gain is unstable, so are all other modes.) The linear stability of such a solution can be examined by writing the perturbed radiation field in the form (cf. Ref. [13] $A = (1 + a)\exp\mathrm{i}(-c_2 z + \phi_0 + \theta)$, where $a = \tilde{a}\cos(\omega_d t)$ and $\theta = \tilde{\theta}\cos(\omega_d t)$ are small and real perturbations. Linearizing Eq. (3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\tilde{a}\\ \tilde{\theta}\end{pmatrix} = \begin{pmatrix}-2-\omega_d^2 & c_1\omega_d^2\\ -2c_2 - c_1\omega_d^2 & -\omega_d^2\end{pmatrix}\begin{pmatrix}\tilde{a}\\ \tilde{\theta}\end{pmatrix}. \quad (5)$$

The eigenvalues for exponential solutions of Eq. (5) are

$$\Lambda = -(1 + \omega_d^2) \pm\sqrt{(1 + \omega_d^2)^2 - (1 + c_1^2)\omega_d^4 - 2(1 + c_1 c_2)\omega_d^2}. \quad (6)$$

In order for the solution to be unstable, Λ must have a positive real part for some ω_d which is possible only if $1 + c_1c_2 < 0$. An extensive parameter search, too detailed to be described here, shows that for realistic FEL parameters, $1 + c_1c_2 > 0$. Thus, the Benjamin–Feir instability does not occur. This is consistent with the results of Refs. [2,3] where a single mode with maximum gain is found to be stable.

Although the principal mode with $\omega_0 = 0$ is linearly stable, a mode with a larger absolute value of ω_0 may be unstable to the Eckhaus instability. In this case, the linearized equations for the perturbations are

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\tilde{a}\\ \tilde{\theta}\end{pmatrix} = \begin{pmatrix} -2a_0^2 - \omega_d^2 - 2ic_1\omega_0\omega_d & c_1a_0\omega_d^2 - 2ia_0\omega_0\omega_d \\ -2c_2a_0 - (c_1\omega_d^2 - 2i\omega_0\omega_d)/a_0 & -\omega_d^2 - 2ic_1\omega_0\omega_d \end{pmatrix}\begin{pmatrix}\tilde{a}\\ \tilde{\theta}\end{pmatrix}, \qquad (7)$$

where $a_0^2 \equiv 1 - \omega_0^2 > 0$. The corresponding eigenvalues are

$$\begin{aligned}\Lambda = {} & \omega_0^2 - 1 - \omega_d^2 - 2ic_1\omega_0\omega_d \\ & \pm \{(\omega_0^2 - 1)^2 - 2c_1\omega_d^2[c_2(1-\omega_0^2) + c_1\omega_d^2/2] \\ & + 4\omega_0^2\omega_d^2 + 4ic_2\omega_0\omega_d(1-\omega_0^2) + 4ic_1\omega_0\omega_d^3\}^{1/2}.\end{aligned} \qquad (8)$$

In the limit of small ω_d, Eq. (8) reduces to

$$\Lambda \xrightarrow{\omega_d \to 0} \begin{cases} -2(1-\omega_0^2) \\ -2\left[1 + c_1c_2 + \dfrac{2\omega_0^2}{\omega_0^2 - 1}(1 + c_2^2)\right]\dfrac{\omega_d^2}{2} + 2i\omega_0\omega_d(c_2 - c_1). \end{cases} \qquad (9)$$

The single mode is unstable if the square bracket of the lower expression in Eq. (9) is negative. Note that this is true for every ω_0 if $1 + c_1c_2 < 0$. However, if $1 + c_1c_2 > 0$, which is generally the case for FELs, the mode will be unstable when $\omega_0^2 > (1 + c_1c_2)/(3 + c_1c_2 + 2c_2^2)$, which can always be satisfied within the range $1 - \omega_0^2 > 0$. Therefore, the Eckhaus instability always exists. In the small $-\omega_d$ limit, it can be shown [4] that the Eckhaus instability is nothing but the phase instability discussed in Ref. [3].

We define the spectral width as $\Delta\omega_d \equiv \langle(\omega_d - \langle\omega_d\rangle)^2\rangle^{1/2}$, where $\langle\omega_d\rangle \equiv \int_{-\infty}^{\infty}\omega_d \tilde{\theta}(\omega_d,z)\,\mathrm{d}\omega_d / \int_{-\infty}^{\infty} \tilde{\theta}(\omega_d,z)\,\mathrm{d}\omega_d$. From Eq. (9), we obtain the solution $\tilde{\theta}(\omega_d,z) = \tilde{\theta}(\omega_d,0)\exp(\Lambda z) \sim \tilde{\theta}_0 \exp[(-h_1\omega_d^2 + ih_2\omega_d)z]$ for the lower eigenvalue, with $\tilde{\theta}_0$, h_1 and h_2 being constants. Thus, we deduce that the spectral half-width decays as $\Delta\omega_d = (2h_1z)^{-1/2} \propto z^{-1/2}$, with z analogous to time, consistent with the scaling reported in Ref. [3]. Note that this scaling applies to a continuum of modes. For discrete modes, the scaling holds until the spectral width is down to about one mode, after which the width decays much faster than $z^{-1/2}$.

To test the analytical predictions discussed above, we report results from a pseudo-spectral code that integrates the GLE. The coefficients c_1 and c_2 are calculated from the parameters of the UCSB FEL [1]: wiggler wave number $k_w = 175\,\mathrm{m}^{-1}$, radiation field wave number $k_s = 15.7\,\mathrm{mm}^{-1}$, wiggler parameter $a_w = 0.3$, plasma frequency $\omega_p = 0.012ck_w$, input electron relativistic factor $\gamma_0 = 7$. Periodic boundary condition is used in t so that A is expanded into a Fourier series of terms with relative frequency (with respect to the reference frequency ω_s) that are multiples of the frequency width $\delta\omega_s$. Note that in the UCSB experiment $\delta\omega_s \approx 20\,\mathrm{MHz}$, $\omega_s \approx 750\,\mathrm{GHz}$. We first consider $\omega_s = \omega_m$, which is stable with $c_1 \approx -0.580$, $c_2 = -0.574$. The simulation is initialized (at $z = 0$) with random small values over all Fourier modes. In this run, 2048 Fourier modes are used initially (with lower resolution for larger z as the spectrum becomes narrower). The unit of time (the period $\approx 31.4\,\mu\mathrm{s}$) is chosen such that the ratio of longitudinal frequency spacing to the principal frequency is $\delta\omega_s/\omega_s = 2\times10^{-3}/75$, which is the same as that of the UCSB experiment [1]. The solid curve in Fig. 1 shows the evolution of the (normalized) spectral width $\Delta\omega$ (in unit of number of

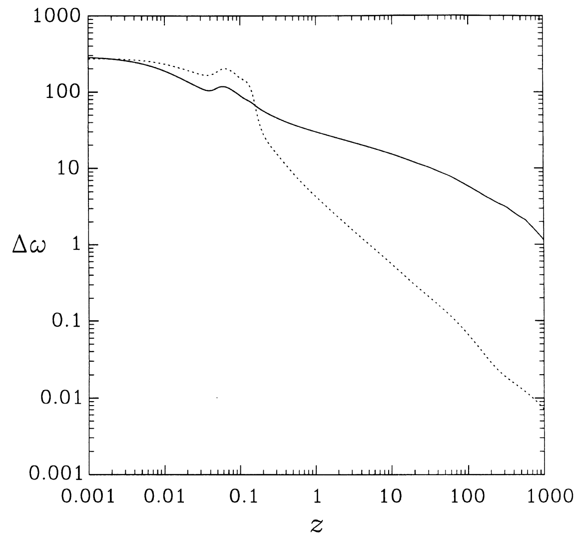

Fig. 1. The evolution along z (μs) of $\Delta\omega$ (solid line) and $\Delta\omega_A$ (dotted line) in simulation of the GLE initialized with small random Fourier amplitudes for $c_1 \approx -0.580$, $c_2 \approx -0.574$. The final state is a single-mode solution with frequency close to the central frequency ω_m.

longitudinal modes) as a function of z (in μs, distance divided by the speed of light). We see that $\Delta\omega$ decays by about two decades when z increases by roughly four decades – consistent with the analytical scaling $\Delta\omega \propto z^{-1/2}$ (which persists till $\Delta\omega$ is reduced to about one mode). The dotted curve is the (normalized) spectral width $\Delta\omega_A$, calculated based on the frequency spectrum of the amplitude of A. We note that $\Delta\omega$ and $\Delta\omega_A$ are of the same order of magnitude for a short, initial interval in z. Thereafter, $\Delta\omega_A$ decays at a much faster rate ($\propto z^{-1}$) than $\Delta\omega$. This implies that the amplitude of A becomes approximately constant well before the solution attains a true single-mode state. In other words, the perturbations in the phase decay much slower than those in the amplitude, as found in Ref. [3]. In quantitative terms, the amplitude becomes roughly constant (when $\Delta\omega_A \sim 1$) at $z \sim 5$ μs. At this stage, $\Delta\omega$ can be viewed as a "macro-mode", consisting of 10–20 eigenmodes that decay eventually to one eigenmode at about $z \sim 1$ ms, which is much longer than the width of the electron pulse (~ 50 μs). We illustrate this further in Fig. 2 which shows radiation field information at $z \approx 2.1$ μs. Plot 2(a) of the Fourier amplitude spectrum shows that it is constituted of nearly 50 modes in the frequency half-width. It is thus not surprising that the time series of the real part of the amplitude A_r has very spiky structure [plot 2(b)] and that the phase ϕ exhibits large temporal variations [plot 2(c)]. However, plot 2(d) shows that the amplitude square $|A|^2$ has already relaxed to a nearly constant value.

Next, we consider a single-mode solution with $\omega_s = 0.977\omega_m$ and $\omega_0 = \mu_i t_0/\alpha_i$ such that $c_1 \approx -0.00944$, $c_2 \approx -4.25$. The system is now subject to the Eckhaus instability although the principal mode is stable. In some cases, the system may respond by shifting the operating mode back to another mode inside the stable range, but this is not what happens in the present case. The upper (lower) curve in Fig. 3 shows $\Delta(\omega)(\Delta\omega_A)$ as a function of z. Both quantities, $\Delta\omega$ and $\Delta\omega_A$, are of the same order of magnitude initially due to the smallness of the initial perturbation that initiates the instability. This is followed by a range in z over which $\Delta\omega$ is more than an order of magnitude larger than $\Delta\omega_A$, although neither quantity is very large. Thus, in the linear regime, the amplitude perturbation grows much slower than the phase perturbation. It is in this sense that the Eckhaus instability is a phase instability, as mentioned above. However, shortly after this linear phase – in the early nonlinear phase – both perturbations grow very rapidly and impulsively, and the system evolves to a nonlinear state with both $\Delta\omega$ and $\Delta\omega_A$ consisting of approximately 40 modes. Although the principal mode (as well as modes in a narrow range around the principal mode) is linearly stable because $1 + c_1c_2 > 0$, the system does not decay to a single-mode state. Instead, it evolves to a nonlinear chaotic state, at least as long as it can be observed numerically [14]. Fig. 4 shows the magnitude A_n of the Fourier modes, indexed by n, of one of such chaotic state, with the parabola showing the shape of the gain curve. We see energy is distributed over a very wide range within the positive gain region, although the power distribution is not as even between the competing modes as the shape of the gain curve might suggest. The power in this chaotic state is very spiky as can be seen in Fig. 5. The width of the spikes vary over a rather large range, with some spikes characterized by widths of the order of

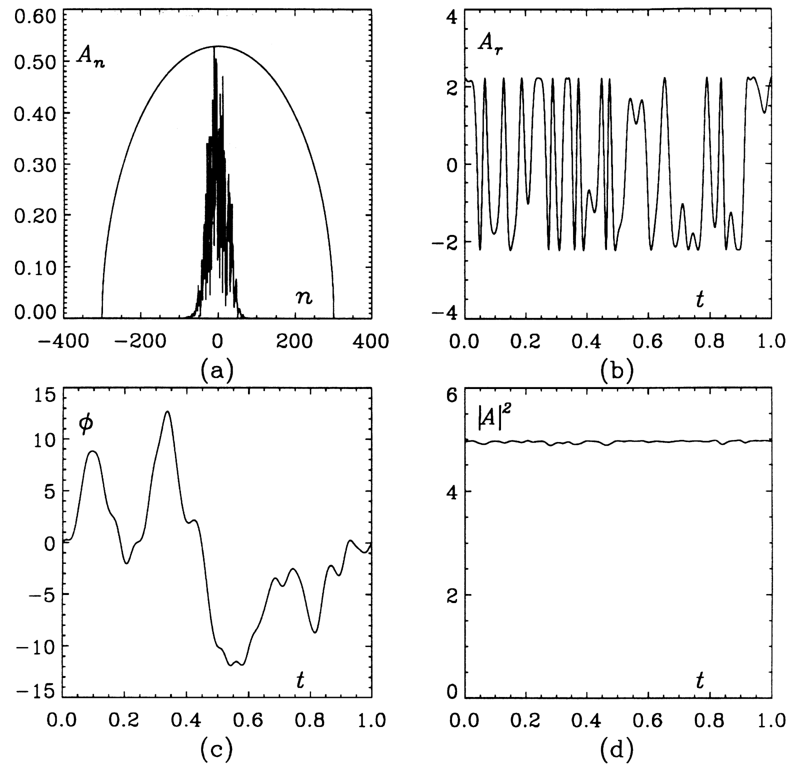

Fig. 2. (a) Fourier amplitude A_n as a function of the Fourier mode number n with the gain parabola plotted in arbitrary units; (b) A_r, the real part of the amplitude; (c) the phase ϕ; and (d) the amplitude square $|A|^2$ of the radiation field plotted as functions of time in one simulation period which is about 31.4 μs at $z \approx 2.1$ μs in the run with parameters of Fig. 1.

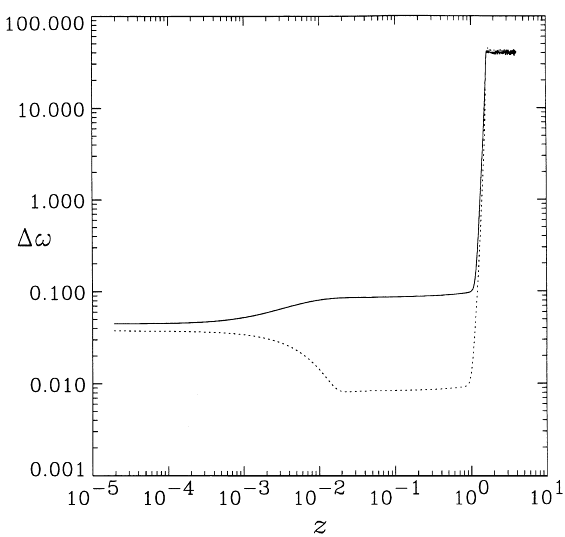

Fig. 3. The evolution along z of $\Delta\omega$ (solid curve) and $\Delta\omega_A$ (dotted curve) for the case of $c_1 \approx -0.00944$, $c_2 \approx -4.25$ starting with an Eckhaus unstable single-mode solution.

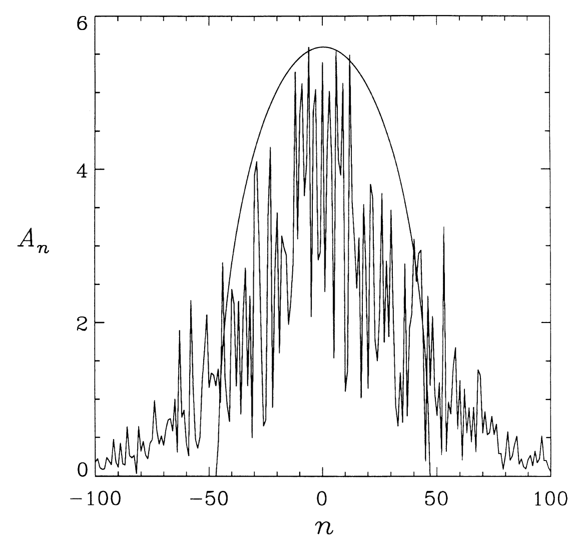

Fig. 4. Amplitude A_n as a function of the Fourier mode number n for the same parameters as Fig. 2 at a z value in the chaotic state. The gain curve, a parabola, is also plotted in arbitrary units.

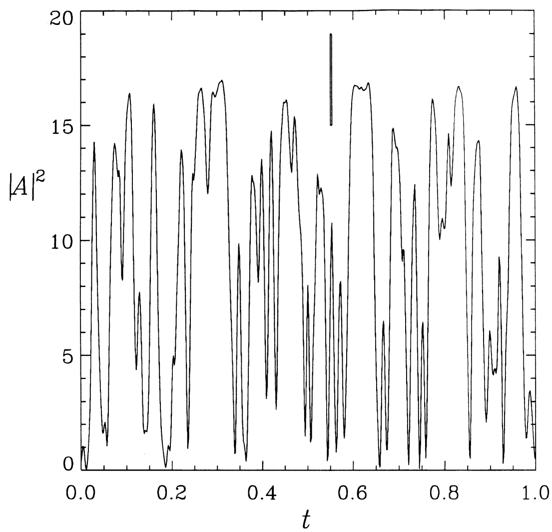

Fig. 5. Amplitude square, $|A|^2$, as a function of time in one simulation period which is about 12.7 μs at the same z as Fig. 4. The width of the box (thick line) in the middle top of the graph represents the value $(-\alpha_i/2\lambda_i)^{1/2}$.

$(-\alpha_i/2\lambda_i)^{1/2}$, as given by the solitary-wave solution of the GLE [5]. (The width of the solitary-wave solution is represented in Fig. 5 as the width of the box (thick line) in the middle–top region.) It thus remains a possibility that some of the spikes observed in experiments [7–9,11] are a result of the Eckhaus instability of a single mode even without the complicating effect of sideband instabilities.

In conclusion, the GLE gives a remarkably precise and detailed description of the dynamical behavior of the radiation field during single-mode operation of an FEL. We have obtained analytical conditions for the Eckhaus instability (known otherwise as the phase instability), verified the analytical results by simulations, and examined conditions under which stable single-mode operation can occur, with specific applications to the UCSB FEL. When the Eckhaus instability does occur, it thwarts single-mode operation, and in the early nonlinear regime, produces a sudden and rapid chaotization of the optical signal. The present application suggests that the Ginzburg–Landau model holds considerable promise as an analytically tractable physical model for the nonlinear radiation field of an FEL.

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