

## A Comparison of Spectral Element and Finite Difference Methods Using Statically Refined Nonconforming Grids for the MHD Island Coalescence Instability Problem

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**Abstract.** A recently developed spectral-element adaptive refinement incompressible magnetohydrodynamic (MHD) code [Rosenberg, Fournier, Fischer, Pouquet, *J. Comp. Phys.* 215, 59-80 (2006)] is applied to simulate the problem of MHD island coalescence instability (MICI) in two dimensions. MICI is a fundamental MHD process that can produce sharp current layers and subsequent reconnection and heating in a high-Lundquist number plasma such as the solar corona [Ng and Bhattacharjee, *Phys. Plasmas*, 5, 4028 (1998)]. Due to the formation of thin current layers, it is highly desirable to use adaptively or statically refined grids to resolve them, and to maintain accuracy at the same time. The output of the spectral-element static adaptive refinement simulations are compared with simulations using a finite difference method on the same refinement grids, and both methods are compared to pseudo-spectral simulations with uniform grids as baselines. It is shown that with the statically refined grids roughly scaling linearly with effective resolution, spectral element runs can maintain accuracy significantly higher than that of the finite difference runs, in some cases achieving close to full spectral accuracy.

### 1. Introduction

In many hydrodynamic or magnetohydrodynamic (MHD) applications in astrophysics or space physics, it is essential that a numerical simulation resolve the development of sharp spatial structures accurately. While pseudo-spectral methods generally can maintain high accuracy, they are mainly applied on more regular geometry and require more uniform grids, which can make it difficult to reach high resolution in order to resolve sharp isolated structures, especially in flows dominated by such structures. Static or adaptive mesh refinement (AMR) methods can put more grid points in and around isolated structures in order to resolve them, but may not achieve similar high-accuracy if low order spatial discretizations are used (Rosenberg et al. 2007). They are particularly useful in bounded

flows, where pseudo-spectral methods are often not optimized. Therefore, it is of great interest to develop numerical schemes that can combine high accuracy and high spatial resolution. Adaptive spectral element methods have the potential to do just that, providing spectral-like accuracy that can be applied efficiently to resolve isolated structures. In this paper, we concentrate on comparing the accuracy of *incompressible* simulation results from a spectral-element-based AMR code (SE) (Rosenberg et al. 2006, 2007) and a finite-difference-based AMR code (FD) (Germaschewski et al. 2006; Bhattacharjee et al. 2005) on an astrophysical problem that requires high spatial resolution as well as high accuracy, the so-called MHD island coalescence instability (MICI) problem (Ng & Bhattacharjee 1998). In order to make meaningful comparisons, we let each code run on essentially the same non-uniform grid (i.e. with the total degrees of freedom in the problem fixed) that is refined a priori in regions of the grid where the current sheets will form during the MICI. A separate pseudo-spectral code (PS), running on uniform grids, is also used to provide a baseline for the comparisons. A more detailed description of the present study can be found in Ng et al. (2008). We will summarize our main results in this paper.

## 2. MHD Island Coalescence Instability

It is well known that a substantial part of our universe is composed of systems of plasmas, ionized gases, and conducting fluids. We consider the representation of magnetohydrodynamics (MHD), as a starting point of discussion. In a dimensionless form, in which all physical quantities are measured by their typical values, the MHD equations generally have dissipation terms involving higher spatial derivatives of the field quantities with strength characterized by the inverse of dimensionless parameters:  $Re = VL/\nu$  is the Reynolds number (with  $V$  a typical flow speed,  $L$  a typical length scale,  $\nu$  the viscosity),  $S = V_A L/\eta$  is the Lundquist number (with  $V_A$  a typical Alfvén speed,  $\eta$  the resistivity). We can immediately see that for most astronomical length scales  $L$ , both  $Re$  and  $S$  are very large numbers such that the dissipation terms in the MHD equations can usually be ignored, (i. e., the ideal MHD equations), except possibly in regions where there exist steep spatial gradients.

Parker (1972, 1979, 1994) has argued for over three decades that current sheets, or tangential discontinuities of the magnetic field, do generally exist in a magnetic equilibrium. In this model, a solar coronal loop is treated as a straight ideal plasma column, bounded by two perfectly conducting end-plates representing the photosphere. The footpoints of the magnetic field in the photosphere are frozen (“line-tied”), and are subjected to slow, random motions that deform the initially uniform magnetic field. Parker (1972) claimed that if a sequence of random footpoint motions are sufficiently complicated, there will be no smooth equilibrium for the plasma to relax to, and tangential discontinuities of the magnetic field (or current sheets) must develop.

Parker’s claim has stimulated considerable debate that continues to this day. For example, the first significant objections to Parker’s claim of non-equilibrium was raised by van Ballegoijen (1985), who argued that smooth equilibria must always exist as long as the footpoint motion is smooth (or continuous). This ar-

gument was based on a reduced form of the MHD equations (referred to hereafter as the RMHD equations) for a low- $\beta$  plasma.

Moreover, based on the same set of RMHD equations, Longcope & Strauss (1994) argued that even when the magnetic equilibrium is unstable, e.g. due to the MICI, it will only relax to a second smooth equilibrium with very thin current layers with thickness less than about  $10^{-7}$  of the large scale.

However, another point of view was raised by Ng & Bhattacharjee (1998), who argued based on a mathematical theorem proved for the RMHD system that for a given fixed footpoint mapping between the two end-plates, there exists only one smooth equilibrium. This means that an unstable equilibrium will relax ideally to a final state with current sheets (tangential discontinuities). This scenario has very different implications than those predicted by Longcope & Strauss (1994), since energy dissipation and other energetic effects can be much stronger for the case with current sheets, than the case with smooth but thin current layers.

Therefore, it is very important to determine which of these two scenarios should actually occur. However, due to the fact that the current layers predicted by Longcope & Strauss (1994) are very thin, it is beyond our computational ability if the full simulation volume is to resolve to the same small scale. To provide a resolution to this problem with current computer architectures, one may need to apply AMR techniques that put more grid points in the regions where distinct structures appear. At the same time, to ensure that any such numerical study is actually representative of the true dynamic solution, one needs to make sure that the numerical scheme used can maintain a reasonably high accuracy.

While the main problem of interest is the three dimensional (3D) problem, for simplicity we will instead simulate the two dimensional (2D) version of the problem for the purpose of this comparative study. Examining this problem in 2D should not affect greatly differences in accuracy among different codes. The 2D MICI problem can be viewed as the limit of the 3D problem with the length of the coronal loop  $L \rightarrow \infty$ . In this limit, it has been shown that current singularities must form for the ideal equations ( $\eta = 0$ ) (Longcope & Strauss 1993). This is beneficial for our present study since we know that there must be specific structures, and we know where they will appear. We can then compare how well these sharp structures are resolved in different schemes. Therefore, we will simulate the set of 2D equations, independent of the third dimension ( $z$ ),

$$\partial_t \Omega + [\phi, \Omega] = [A, J] + \nu \nabla_{\perp}^2 \Omega, \quad (1)$$

$$\partial_t A + [\phi, A] = \eta \nabla_{\perp}^2 A, \quad (2)$$

where  $A$  is the flux function so that the magnetic field is expressed as  $\mathbf{b} = \hat{\mathbf{z}} + \nabla_{\perp} A \times \hat{\mathbf{z}}$ ,  $\phi$  is the stream function so that the velocity field is expressed as  $\mathbf{u} = \nabla_{\perp} \phi \times \hat{\mathbf{z}}$ ,  $\Omega = -\nabla_{\perp}^2 \phi$  is the  $z$ -component of the vorticity,  $J = -\nabla_{\perp}^2 A$  is the  $z$ -component of the current density, and  $[\phi, A] = \partial_y \phi \partial_x A - \partial_y A \partial_x \phi$ .

### 3. Methods and Results

The SE method evolves MHD equations in time, as part of the *Geophysics/Astrophysics Spectral Element Adaptive Refinement (GASpAR)* code, and has been

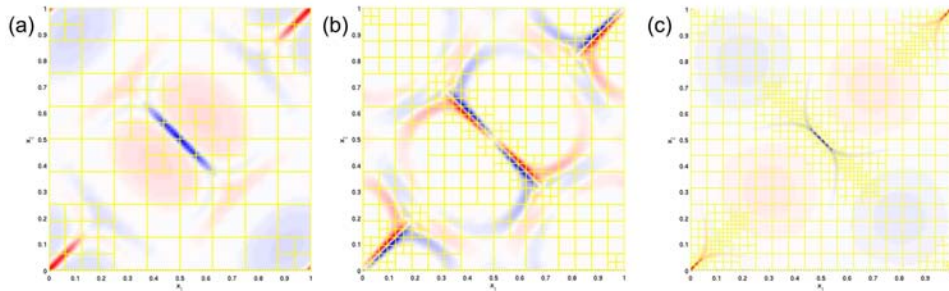


Figure 1. (a) The grid used in the  $\mathcal{R}_{\text{eq}} = 128$  case. The background is the color contour plot of the current density  $J$  at  $t = 1.3$ . (b) The grid used in the  $\mathcal{R}_{\text{eq}} = 256$  case, with the plot of the vorticity  $\Omega$  at  $t = 1.3$ . (c) The grid used in the  $\mathcal{R}_{\text{eq}} = 512$  case, with the plot of  $J$  at  $t = 0.93$ .

described in detail in Rosenberg et al. (2006, 2007). The code has an adaptive mesh capability but it is not used in this work. Instead, the nonconforming grid is constructed initially by turning off the refinement criteria, and selecting the elements we want to refine explicitly. The nonconforming grid is then used in a static configuration throughout the simulation. *Nonconforming* in this context means that there are at most two child elements adjacent to a coarser neighbor; an element is refined by dividing it isotropically into  $2 \times 2$  child elements, each of which contains the same polynomial order as the parent.

The FD method used in this study is based on the *Magnetic Reconnection Code (MRC)*, i.e., a suite of codes (Germaschewski et al. 2006; Bhattacharjee et al. 2005) which integrate various reduced and extended fluid models of plasma flows. In this paper, the 2D AMR version of the code integrating the equations of RMHD has been used.

For the purpose of the present study, the dynamic refinement capability of the AMR code is turned off so that we can use the same grids that were also used in SE. This was done because it is difficult to set refinement methods and criteria to be the same in both the FD and SE codes. Using static refinement provides us with nonconforming grids over which we can exert complete control of the number of degrees of freedom. Comparison of refinement criteria in the SE and FD codes and the effect on the resulting dynamics is left for future work.

The PS code is based on fast Fourier transform (FFT) on a 2D bi-periodic domain. It is de-aliased by the standard 2/3 rule. The nonlinear term is calculated in the physical space on a uniform grid of collocation points. A second order predictor-corrector method is used for time integration. For this study, results from SE and FD methods for the effective resolution  $\mathcal{R}_{\text{eq}}$  case are compared with those from the PS code with  $\mathcal{R}_{\text{eq}} \times \mathcal{R}_{\text{eq}}$  collocation points. The results from the PS code are themselves checked with PS runs with higher resolutions, up to  $2048 \times 2048$ , which confirm that the results with the original resolution are already well-resolved.

Fig. 1 shows refinement grids used in the SE and FD runs for equivalent resolution  $\mathcal{R}_{\text{eq}} = 128$  ( $\eta = \nu = 2 \times 10^{-3}$ ) in (a),  $\mathcal{R}_{\text{eq}} = 256$  ( $\eta = \nu = 10^{-3}$ ) in (b), and  $\mathcal{R}_{\text{eq}} = 512$  ( $\eta = \nu = 3 \times 10^{-4}$ ) in (c), where  $\mathcal{R}_{\text{eq}} \times \mathcal{R}_{\text{eq}}$  is the corresponding uniform grid. Color contour plots of  $\Omega$  and  $J$  are also shown in Fig. 1 (red for the

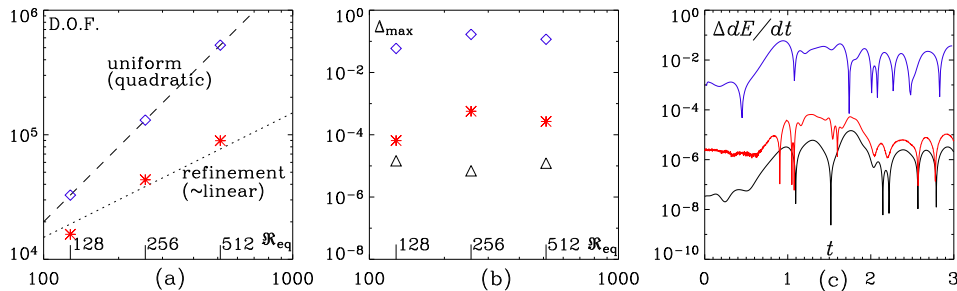


Figure 2. (a) D.o.f. of the refinement grids at the different  $\mathcal{R}_{\text{eq}}$  and dissipation levels, in red asterisks, and those using uniform grids, in blue diamonds. (b) Maximum fractional error over the duration of simulation of  $\Delta \dot{E}$  (black triangles for PS, red asterisks for SE, blue diamonds for FD) for the three  $\mathcal{R}_{\text{eq}}$  cases. (c)  $\Delta \dot{E}$  for the  $\mathcal{R}_{\text{eq}} = 128$  case. Black curve is for the PS run, red curve is for the SE run, and blue curve is for the FD run.

positive end and blue for the negative end). Refinement is needed mainly along sharp vorticity and current layers. Therefore, the d.o.f. of these grids require only a roughly linear scaling with  $\mathcal{R}_{\text{eq}}$ , whereas uniform grids follow a scaling of  $\mathcal{R}_{\text{eq}}^2$  as they should, as shown in Fig. 2 (a). Since as  $\mathcal{R}_{\text{eq}}$  increases, the difference between the two scalings can be very large, using adaptive refinement has the potential of providing considerable savings in memory and/or CPU usage, if the linear scaling of the d.o.f. in these refinement grids continues to hold for even larger  $\mathcal{R}_{\text{eq}}$ . Many other quantities from these runs are compared for accuracy Ng et al. (2008). We show one representative quantity in Fig. 2 (b) and (c), the fractional error  $\Delta \dot{E}$  between the two sides of the equation of conservation of energy,

$$\frac{dE}{dt} \equiv \frac{1}{2} \frac{d[\langle u^2 + b^2 \rangle]}{dt} = -\nu \langle \Omega^2 \rangle - \eta \langle J^2 \rangle, \quad (3)$$

where for any function  $\phi$ ,  $\langle \phi \rangle \equiv \int \phi d\mathbf{x}^2$ . In Fig. 2 (b), maximum  $\Delta \dot{E}$  over the duration of the runs are shown for different levels of  $\mathcal{R}_{\text{eq}}$ .  $\Delta \dot{E}$  as a function of  $t$  is plotted in (c) for each code for the  $\mathcal{R}_{\text{eq}} = 128$  case. We see that while PS runs (black triangles) have the smallest errors as expected, the errors in the SE runs (red asterisks) are also quite low. In fact, the errors in the SE runs sometimes approach the error levels of the PS runs. We also see that maximum errors in the FD runs (blue diamonds) reach a rather high level, close to 10% at times, and have a large separation from the other two sets of data. Note that the three cases are for three different dissipation levels and so the error is not supposed to get smaller for the higher resolution case. While more comparative study on these codes still need to be done in terms of convergence and efficiency, we have shown the potential advantage of SEMs in terms of much higher accuracy. This is especially important in the 3D Parker model, since one would need to draw a *definite* conclusion as to whether true current sheets form in ideal MHD, or whether only thin but smooth current layers form. A conclusion based on a method with much higher accuracy should be more reliable, since subtle errors in runs of the 3D Parker problem may change the dynamics in favor of one or the other scenarios.

#### 4. Discussion and Conclusion

It is worth noting that at the present stage, the spectral element method described in this paper is more costly in terms of computational time than the FD and PS methods with which we compare our results, the latter being optimal for periodic boundary conditions. We have not attempted to compare the codes in terms of performance because to do so impartially, while a very interesting and perhaps even necessary research topic, is well beyond the scope of this paper. Instead, we concentrate on comparing accuracy obtained by the two schemes, mainly to answer the question of whether using SE scheme has any advantage that deserves putting more effort into further development for the Parker problem. We believe the results of this paper have shown that, indeed, the SE method has substantial advantages and needs to be investigated further.

We have shown that using statically refined grids with DOF of roughly linear scaling, SE method can produce results with high accuracy that can sometimes even approach the spectral accuracy of the PS method for the 2D MICI problem. This can be very helpful in the investigation of the 3D MICI problem in the ideal limit, which will require adaptive grids that can resolve distinct features. The higher accuracy of the SE method can potentially provide a reliable definitive answer to the important question of whether a true current sheet forms in the Parker problem of solar and stellar coronal heating.

Our main conclusion that SE methods can produce simulations with accuracy somewhat in between PS and FD methods is not surprising in itself. However, our results yield important quantitative information in the context of statically refined grids in problems with distinct spatial structures.

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