

# Rutherford Scattering Made Simple

Chung-Sang Ng

*Physics Department, Auburn University , Auburn, AL 36849*

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Rutherford scattering experiment<sup>1</sup> is so important that it is seldom not mentioned in any introductory modern physics course. For those non-calculus classes, it would be hard to discuss many details, such as the derivation of the differential cross section, about it. Even so, some quantitative discussion on how the scattering angle depends on the impact parameter, based on the classical Coulomb model, will give students a better physical picture.

The more complete approach that derive the hyperbolic trajectory by solving the Newton's second law, using differential equations<sup>2</sup>, is obviously not suitable for this purpose. The derivation using the impulse-momentum relation<sup>3,4</sup>, without knowing the exact form of the trajectory, is neat, but still require integration. This may be hard to teach to a non-calculus class.

The method that Rutherford himself and others used<sup>1,4</sup> is indeed non-calculus. It assumes the trajectory to be a branch of a hyperbola. Students should not be hard to imagine that the trajectory is hyperbola-like, because of the scattering nature of the problem and that the forces between the particles is repulsive. However, this method also assumes the heavy nucleus to be at the outer focus of the hyperbola. This is not so easy to understand. Moreover, it has to use the properties of a hyperbola, e. g., the distant from the center to the vertex of a branch is equal to the projection of the distance from the center to a focus on an asymptotes. Students may have a hard time understanding this without a long review of a hyperbola.

Here I try to further simplify the last approach. Consider a charged particle with mass  $m$ , charge  $ze$ , coming in from the left with initial speed  $v_0$ , and initial direction along a

line with a distance  $b$  (the impact parameter) to the nucleus, see Fig. 1. The nucleus with charge  $Ze$  and mass  $M$  is at rest if we use the relative frame or assume that  $M \gg m$ . Let the particle, scattered by the nucleus, go out asymptotically along a line making an angle  $\theta$  with the incoming line. Let the two lines intersect at point  $O$ . Assume that the trajectory of the particle is a hyperbola-like curve in the sense that it has two asymptotes, the incoming and outgoing lines, and that there is another branch of the “hyperbola” symmetric to it about the center  $O$ . We make a natural assumption that this branch is the trajectory of another particle with the same mass  $m$ , but opposite charge  $-ze$ , coming in from the right with the same speed  $v_0$ . These two trajectories being symmetric suggests that all physical parameters should have the same magnitude for both cases. However, since this particle is obviously attracted by the nucleus, it should have negative charge.

These two trajectories also must be symmetric about the central line passing through the center  $O$  and the position of the nucleus. This is due to the symmetry of the problem. If the two particles were to come from the original outgoing line with the same speed, they would pass through exactly the same trajectories and would go out along the original incoming line respectively. Therefore, the two vertices  $V_{\pm}$  of the two branches must be lying on this central line, and the directions of the velocities  $\mathbf{v}_{\pm}$  of two particles at  $V_{\pm}$  must be perpendicular to it. Using the law of conservation of energy, we have

$$\frac{1}{2}mv_0^2 = \pm \frac{zZe^2}{4\pi\epsilon_0 L_{\pm}} + \frac{1}{2}mv_{\pm}^2, \quad (1)$$

where  $L_{\pm}$  are the distances from the two vertices  $V_{\pm}$  to the position of the nucleus. By the law of conservation of angular momentum, we have

$$mv_0b = mv_{\pm}L_{\pm}. \quad (2)$$

FIGURES

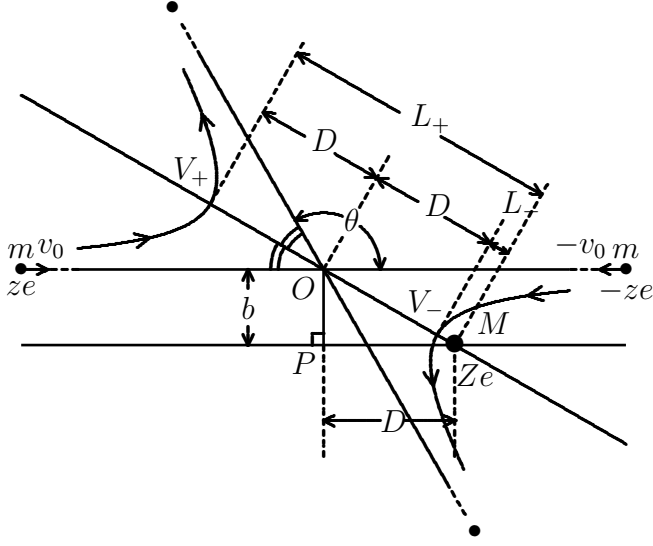


FIG. 1. Schematic diagram of Rutherford scattering. See text for details.

Combining Eqs. (1) and (2), we have

$$L_{\pm}^2 \mp 2DL_{\pm} - b^2 = 0, \quad (3)$$

where

$$2D \equiv \frac{zZe^2}{4\pi\epsilon_0} \bigg/ \frac{1}{2}mv_0^2, \quad (4)$$

is the closest impact distance. Solutions of Eq. (3) are,

$$L_{\pm} = \sqrt{D^2 + b^2} \pm D, \quad (5)$$

where we have discarded negative solutions by physical requirement. By assumption, the distances from the center  $O$  to the two vertices  $V_{\pm}$  are the same, i. e.,  $\overline{OV_+} = \overline{OV_-}$ . Therefore, we have

$$L_{\pm} = \overline{OM} \pm \overline{OV_+}, \quad (6)$$

where  $\overline{OM}$  is the distance from  $O$  to the position of the nucleus.

By comparing Eqs. (5) and (6), we get

$$\overline{OM} = \sqrt{D^2 + b^2}, \quad \overline{OV_+} = \overline{OV_-} = D.$$

So we have  $\overline{PM} = D$ , by Pythagoras theorem, where  $\overline{PM}$  is the projection of the distance  $\overline{OM}$  on a horizontal line. By elementary geometry, it is not hard to show that  $\angle POM = \theta/2$ . Finally, by the definition of the tangent function, we get the relationship between the scattering angle  $\theta$  and the impact parameter  $b$ ,

$$\tan \frac{\theta}{2} = \frac{D}{b}. \quad (7)$$

The above derivation does not use the less obvious assumption that the nucleus is at the outer focus of the hyperbola. However, we can now check that this is indeed so since the projection of  $\overline{OM}$  on one of the asymptotes equals to the distance from the center to either one of the vertices.

From this derivation, we can easily deduce a fact usually not mentioned: the scattering pattern is independent of the sign of the charge. A scattering experiment using anti- $\alpha$ -particles would give the same scattering pattern as that using ordinary  $\alpha$ -particles, if all the physics could really be described by a classical Coulomb model alone.

Since we still have Fig. 1 here, let me bring up another point usually not discussed. Instead of solving the scattering angle  $\theta$  by an inverse tangent operation, using Eq. (7), we can easily find it by graphical method. Teaching this method will give a clearer physical picture to many students, even they understand the mathematics. The graphical method is simple. First, draw a horizontal incoming line with a distance  $b$  to the nucleus. Second, draw a vertical line with a distance  $D$  in front of the nucleus, with  $D$  calculated by Eq. (4). Let the intersection of these two lines be the center  $O$ . Then, draw the central line passing through the nucleus and  $O$ . Finally, draw another line passing through  $O$  that makes the same angle with the central line as that made by the incoming line. This is the outgoing line. From this line, we can easily measure the scattering angle  $\theta$ . This graphical method is very useful when drawing more than two trajectories on the same graph, especially with the same initial velocity  $v_0$  but different impact parameter  $b$ .

There is yet another easy rule to help memorizing this graphical method, and thus the formula (7): the particle is scattered with the same angle as that scattered by a non-charged elastic hard sphere centered at the position of the nucleus, with radius  $(D^2 + b^2)^{1/2}$ . By this rule, we see that the particles are scattered as if by a hard sphere with radius  $D$ , for the cases with  $b \ll D$ . These cases give large scattering angles and show that a nucleus exists. However, as the impact parameter becomes larger, this “hard sphere” also expands by nearly the same rate. So the incoming particles are always being scattered. From this we can see the long range nature of the Coulomb potential.

In conclusion, the above derivation of the relationship between the scattering angle and the impact parameter uses a more natural assumption and does not require advanced mathematics. However, it still gives a lot of important physics about the Rutherford scattering.

## REFERENCES

- [1] E. Rutherford, “Scattering of  $\alpha$ - and  $\beta$ - particles by matter, and the structure of the atom,” *Phil. Mag.* **21** 669–688 (1911). Part of this paper was reproduced in the following text book: C. Kittel, W. D. Knight, and M. A. Ruderman, *Mechanics – Berkeley Physics Course, Volume 1* (Educational Services Incorporated), pp. 459–467.
- [2] For examples, R. M. Eisberg, *Fundamentals of Modern Physics* (Wiley, New York, 1961), pp. 99–104; H. A. Enge, M. R. Wehr, and J. A. Richards *Introduction to Atomic Physics* (Addison–Wesley, Reading, Massachusetts, 1981), pp. 90–93.
- [3] M. M. Gordon, “Concerning the Rutherford scattering formula” *Am. J. Phys.* **23** 247–248 (1955).
- [4] For example, U. Fano, and L. Fano, *Basic Physics of Atoms and Molecules* (Wiley, New York, 1959), pp. 344–348.