Curved Spacetime

In Newtonian mechanics we describe curves parametrically by assuming time is an independent variable increasing at a constant rate. In this way the geometry is three-dimensional and the location of an object at some moment in time is $\vec{r}(t)$ where the components of $\vec{r}(t)$ depend upon the coordinate system used. In GR time is not an independent variable and so must be considered one component of the four components required to specify the location of an object. For this reason a system of four-vectors is used such that each point in space is labeled by four coordinates: x^i, $i = 0, 1, 2, 3$. It is important to note that the superscript labels the component of the vector and does not represent a power, etc. Conventional usage puts $x^0 = t$, the time coordinate leaving x^1, x^2 and x^3 as the space coordinates. This four-dimensional space is called 'spacetime'. The path of a particle in spacetime is called its 'world line'.

An invariant quantity along a particle's path is the *proper time*, $\vec{\tau}$. This is the time indicated by a standard clock moving with the particle. As a particle moves along its world line the arc length between points x^μ and $x^\mu + \Delta x^\mu$ is written

$$\Delta\tau^2 = \Delta t^2 - |\Delta\vec{r}|^2/c^2 \tag{1}$$

or, in cartesian coordinates,

$$\Delta\tau^2 = \Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)/c^2. \tag{2}$$

Notice that it is possible to find two points in spacetime such that $\Delta\tau < 0$. This implies that $|\Delta\vec{r}| > c\Delta t$ or $|\Delta\vec{r}|/\Delta t > c$. A particle moving between these two points would move faster than light: an impossibility. So there is no exhange of information possible between these two points.

The separation between points in spacetime can be:

spacelike $\Delta\tau^2 > 0$ (within the light cone)
null separation $\Delta\tau^2 = 0$ (the light cone)
timelike $\Delta\tau^2 < 0$ (outside the light cone)

Basic Linear Algebra of Four-Vectors

Let $\hat{e}_0, \hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ be the basis set in our four-vector space. Then any vector $\vec{a}$ can be written

$$\vec{a} = a^0\hat{e}_0 + a^1\hat{e}_1 + a^2\hat{e}_2 + a^3\hat{e}_3. \tag{3}$$

or, equivalently,

$$\vec{a} = \sum_{\alpha=0}^{3} a^\alpha \hat{e}_\alpha. \tag{4}$$

Here we inroduce the *summation convention* in which we infer an implied sum any time an index is repeated. Using this, equation (4) can be written

$$\vec{a} = a^\alpha \hat{e}_\alpha. \tag{5}$$

The Scalar Product and the Metric Tensor

The inner product of two four-vectors is given by

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (a^\alpha\hat{e}_\alpha)\cdot(b^\beta\hat{e}_\beta),\\ &= (\hat{e}_\alpha\cdot\hat{e}_\beta)a^\alpha b^\beta.\end{aligned} \tag{6}$$

More compactly,

$$\vec{a}\cdot\vec{b} = g_{\alpha\beta}a^\alpha b^\beta \tag{7}$$

where $g_{\alpha\beta}$ is called the metric tensor.

In spacetime the proper time interval can be written as

$$\Delta\tau^2 = g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta \tag{8}$$

with

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/c^2 & 0 & 0 \\ 0 & 0 & -1/c^2 & 0 \\ 0 & 0 & 0 & -1/c^2 \end{pmatrix}. \tag{9}$$

One can show that the normal associative and distributive laws of inner products hold with this definition[1].

Geodesics in Spacetime

What path does a body follow if no non-gravitational forces act upon it? The answer in GR is that the world line of a free test particle[2] between two time-like points is one which minimizes the proper time between the points.

Writing the separation between two events, A and B as τ_{AB} then

$$\begin{aligned}\tau_{AB} &= \int_A^B \sqrt{g_{ij}\Delta x^i \Delta x^j} \\ &\int_A^B \sqrt{g_{ij}(x^k(\tau))\frac{dx^i(\tau)}{d\tau}\frac{dx^j(\tau)}{d\tau}}\end{aligned} \tag{10}$$

In practice, one evaluates the variation of equation (10) and solves for extremal values as the path is varied. The result is analogous to the principle of least action in Newtonian mechanics. There, the path of motion is one that minimizes the action. When this principle is applied to the action integral Lagrange's equations result. These extremal paths are called geodesics. The path of interest here is the path producing the minimum in the proper time difference between the events. The world lines of of bodies not subjected to non-gravitational forces are space-like geodesics. The world lines of light rays are null geodesics.

Gauss' Theorema Egregium

Gauss showed that the curvature of a general surface is[3]

$$\begin{aligned}K = \frac{1}{2g_{11}g_{22}}\Big\{-\frac{\partial^2 g_{11}}{\partial(x^2)^2} - \frac{\partial^2 g_{22}}{\partial(x^1)^2} + \frac{1}{2g_{11}}\Big[\frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1} + \Big(\frac{\partial g_{11}}{\partial x^2}\Big)^2\Big] \\ + \frac{1}{2g_{22}}\Big[\frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2} + \Big(\frac{\partial g_{22}}{\partial x^1}\Big)^2\Big]\Big\}\end{aligned} \tag{11}$$

This is an impressive and imposing equation indeed. However, by applying it to known surfaces it becomes a bit less threatening. As an example, consider a spherical surface, $r = R$. For this geometry let $x^1 = \theta$ and $x^2 = \phi$. Recall, for an arc Δs on the surface, $\Delta s^2 = R^2(\Delta\theta)^2 + R^2 \, sin^2\,\theta(\Delta\phi)^2 = R^2(\Delta x^1)^2 + R^2 \, sin^2\, x^1(\Delta x^2)^2$. From this it is easy to infer the metric tensor:

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \, sin^2\, x^1. \end{pmatrix} \tag{12}$$

[1] Hartle, Chapter 5
[2] A test particle is so small that it has no effect on spacetime but rather it moves in response to the gravitational effects of other masses present.
[3] M. V. Berry, Principles of Cosmology and Gravity, IOP press, 1993

From this we can begin to calculate the terms in equation (11). First, notice that

$$\frac{\partial g_{11}}{\partial x^1} = \frac{\partial g_{11}}{\partial x^2} = \frac{\partial g_{22}}{\partial x^2} = 0;$$

while

$$\frac{\partial g_{22}}{\partial x^1} = 2R^2\, sin\, x^1\, cos\, x^1$$
$$\frac{\partial^2 g_{22}}{\partial (x^1)^2} = 2R^2 (cos^2 x^1 - sin^2 x^1).$$

Inserting these into equation (11) and simplifying the algebra gives

$$K = \frac{1}{R^2} \tag{12}$$

for the curvature of a spherical surface, radius R.

Static Spacetime Curvature

Consider a symmetric, three dimensional space with constant curvature K. Following Berry[3], we shall describe this space using spherical coordinates. We can write the element of arc length on the surface at r as

$$\Delta s^2 = r^2 \Delta\theta^2 + r^2\, sin^2\, \theta \Delta\phi^2.$$

By definition, the area of this surface is $4\pi r^2$. Because this space has curvature, r is not the proper radius of the sphere because the space is curved. A tape measure stretched from the origin to the surface of a sphere would not be r. We can allow for this by writing the complete metric as

$$\Delta s^2 = f(r)\Delta r^2 = r^2 \Delta\theta^2 + r^2\, sin^2\, \theta \Delta\phi^2 \tag{13}$$

where $\sqrt{f(r)}$ is the proper distance between (r, θ, ϕ) and $(r + \Delta r, \theta, \phi)$.

We can find an expression for $f(r)$ using the fact that all geodesics have the same curfvature, K. This allows us to look at the simple case of the geodesic at the equator where $\theta = \pi/2$ and $\Delta\theta = 0$. There,

$$\Delta s^2 = f(r)\Delta r^2 + r^2 \Delta\phi^2. \tag{14}$$

By inspection, the metric for this surface is

$$g_{ij} = \begin{pmatrix} f(x^1) & 0 \\ 0 & (x^1)^2 \end{pmatrix}. \tag{15}$$

Using Gauss' formula (equation (11) above) one can show that

$$K = \frac{df(x^1)/dx^1}{2f^2(x^1)x^1}. \tag{16}$$

Since K is a constant over the whole case, equation (16) can be integrated by noting that

$$\frac{df(x^1)/dx^1}{f^2(x^1)x^1} = -\frac{d}{dx^1}\Big(\frac{1}{f(x^1)}$$

and so

$$-\frac{d}{dx^1}\Big(\frac{1}{f(x^1)}\Big) = 2kx^1.$$

This is easily integrated to give

$$f(x^1) = \frac{1}{C - K(x^1)^2}. \tag{17}$$

Where C is a constant of integration. C can be evaluated by noting that when $K = 1$, $f = 1$ and so C=1. Thus our metric for this case is

$$\Delta s^2 = \frac{\Delta r^2}{1 - Kr^2} + r^2 \Delta\phi^2. \tag{18}$$

The proper radius of a sphere at r is

$$\begin{aligned} a =& \int_0^r \frac{dr}{\sqrt{1 - Kr^2}} \\ =& \frac{1}{\sqrt{K}} \arcsin(\sqrt{K}r) \end{aligned} \tag{19}$$

or

$$r = \frac{1}{\sqrt{K}}\, sin(a\sqrt{K}). \tag{20}$$

And, since the area $A = 4\pi r^2$ the area is

$$A = \frac{4\pi}{K}\, sin^2(a\sqrt{K}). \tag{21}$$

Notice that for small spheres, as $a \ll 1/\sqrt{K}$, A approaches $4\pi a^2$ and reaches a maximum value of $4\pi/K$ when $a = \pi/(2\sqrt{K})$ and then decreases to zero as $a = \pi/\sqrt{K}$. This can be viewed as the proper radius on a sphere starting at the North pole, maximizing at the equator and then diminishing to zero at the South pole.

Homework

1) Let x, y and z be Cartesian coordinates in ordinary three-dimensional space and let x^1, x^2 and x^3 be a set of generalized coordinates, not necessarily orthogonal. We may write

$$x = x(x^1, x^2, x^3); \quad y = y(x^1, x^2, x^3); \quad z = z(x^1, x^2, x^3). \tag{$hw1$}$$

Derive the transformation between differences Δx, etc. and the Δx^α using the ordinary chain rule. Organize the result and write the arc element $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ in terms of the Δx^α and show that Δs^2 can be written in the form $\Delta s^2 = g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta$. Write out the expression for $g_{\alpha\beta}$ explicitly.

Cartesian plane: let $x^1 = x$ and $y^1 = y$ and show that

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{$hw2$}$$

Polar coordinates in the plane: let $x^1 = r$ and $x^2 = \phi$, and show that

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{$hw3$}$$

2) Consider two events on the world line of a particle moving with speed $v = |\Delta\vec{r}|/\Delta t$. Write the invariant interval $\Delta\tau^2$ and solve for the relation between Δt and $\Delta\tau$. The result should be the relationship between proper time and clock time in special relativity (time dilation).

3) Consider a frame f' moving at a velocity v relative to an inertial frame f (assume the motion is in the of direction $\hat{x}^1$. Two events occur:

event 1: the origins of f and f' coincide;
event 2: a point in f along $\hat{x}^1$ marked L coincides with the origin of f'.

Write down the invariant interval in each coordinate system and deduce the Lorentz contraction formula.

4) Explore the result one obtains for equations (20) and (21) if K is negative. That is, let $K = -|K|$ and rederive (20) and (21). Does the surface area in this case change faster or slower than flat space ($K = 1$)?