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Arthur Smith and the basic rules of calculus

1. Introduction

In his manuscript entitled “Proof of the atmospheric greenhouse effect” Arthur Smith (2008, see http://arxiv.org/abs/0802.4324) introduced effective quantities by averaging over the entire globe (see his Eqs. (7) to (9)). Obviously, there is a violation of basic rules of calculus. This fact is explained here.

If we define such effective values by averaging over the entire globe, we will obtain

$$\left\langle \Psi \right\rangle = \frac{1}{4\,\pi}\int_{\Omega} \Psi \; d\Omega \quad . \tag{1}$$

Here, Ψ is an arbitrary variable, Ω is the solid angle (for a sphere $\Omega = 4\,\pi$) and $d\Omega$ is the differential solid angle. Note the radius of the globe does not play a role. For $\Psi = T^4$ we immediately obtain

$$\left\langle T^4 \right\rangle = \frac{1}{4\,\pi}\int_{\Omega} T^4 \; d\Omega \quad . \tag{2}$$

This equation is identical with Eq. (7) of Smith (2008). For the radiative emission of energy we may write

$$\sigma \left\langle \varepsilon\, T^4 \right\rangle = \frac{\sigma}{4\,\pi}\int_{\Omega} \varepsilon\, T^4 \; d\Omega \tag{3}$$

or

$$\left\langle \varepsilon\, T^4 \right\rangle = \frac{1}{4\,\pi}\int_{\Omega} \varepsilon\, T^4 \; d\Omega \tag{4}$$

Following Smith (2008, Eq. (8)) this expression can be rearranged by

$$\left\langle \varepsilon \right\rangle = \frac{1}{4\,\pi \left\langle T^4 \right\rangle}\int_{\Omega} \varepsilon\, T^4 \; d\Omega$$

This is obviously wrong because it would mean that $\left\langle \varepsilon\, T^4 \right\rangle = \left\langle \varepsilon \right\rangle \left\langle T^4 \right\rangle$. The correct result is

$$\left\langle \varepsilon\, T^4 \right\rangle = \frac{1}{4\,\pi}\int_{\Omega} \varepsilon\, T^4 \; d\Omega \neq \left\langle \varepsilon \right\rangle \left\langle T^4 \right\rangle = \frac{1}{4\,\pi}\int_{\Omega} \varepsilon \; d\Omega \, \frac{1}{4\,\pi}\int_{\Omega} T^4 \; d\Omega \quad . \tag{5}$$

In his response (http://arthur.shumwaysmith.com/life/content/why_are_some_people_so_easily_confused), Smith stated I am confused. Therefore, I explained it further. Equation (7) of Smith reads:

$$T_{eff}(t)^4 = \frac{1}{4\,\pi\,r^2}\int T(x,t)^4\,dx$$

According to Smith (2008), $T_{eff}(t)^4$ is defined as an average over the planetary surface. Note that even this notation is awkward because the quantity planetary-averaged is $T(x,t)^4$, i.e., it should write $\left(T(t)^4\right)_{eff}$. Smith claimed that his Eq. (8),

$$\varepsilon_{eff}(t) = \frac{1}{4\,\pi\,r^2\;T_{eff}(t)^4}\int \varepsilon(x,t)\,T(x,t)^4\,dx \quad ,$$

should be considered as a definition of $\varepsilon_{eff}(t)$. If this is true, then this quantity cannot be considered as an average over the planetary surface as Smith (2008) stated in the sentence directly followed after his Eq. (6). A planetary-averaged $\varepsilon(x,t)$ is given by

$$\varepsilon_{eff}(t) = \frac{1}{4\,\pi\,r^2}\int \varepsilon(x,t)\,dx$$

Furthermore, averaging the product $\varepsilon(x,t)\,T(x,t)^4$ over the planetary surface yields

$$\left(\varepsilon(t)\,T(t)^4\right)_{eff} = \frac{1}{4\,\pi\,r^2}\int \varepsilon(x,t)\,T(x,t)^4\,dx$$

Dividing this equation by $T_{eff}(t)^4$ provides

$$\{\varepsilon(t)\} = \frac{\left(\varepsilon(t)\,T(t)^4\right)_{eff}}{T_{eff}(x,t)^4} = \frac{\frac{1}{4\,\pi\,r^2}\int \varepsilon(x,t)\,T(x,t)^4\,dx}{\frac{1}{4\,\pi\,r^2}\int T(x,t)^4\,dx} = \frac{\int \varepsilon(x,t)\,T(x,t)^4\,dx}{\int T(x,t)^4\,dx} \neq \frac{1}{4\,\pi\,r^2}\int \varepsilon(x,t)\,dx$$

This means that Smith is wrong because $\{\varepsilon(t)\} \neq \varepsilon_{eff}(t)$. I wonder who confuse is.

2. Averaging in Turbulence

Here is another instance that underlines the absurdity of Smith's (2008) suggestion.

The time average of an arbitrary time-dependent quantity $f(t)$ is given by

$$\overline{f} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, dt \quad , \tag{6}$$

If we express $f(t)$ by $f(t) = \overline{f} + f'(t)$ the definition (6) yields

$$\overline{f} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(\overline{f} + f'(t)\right) dt = \overline{f} + \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f'(t)\, dt \quad . \tag{7}$$

From this equation one can infer that

$$\overline{f'} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(f'(t)\right) dt = 0 \quad . \tag{8}$$

According to these equations the time average of the product of two arbitrary time-dependent quantities $f(t)$ and $g(t)$ is given by

$$\overline{f\, g} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, g(t)\, dt \quad . \tag{9}$$

Expressing $f(t) = \overline{f} + f'(t)$ and $g(t) = \overline{g} + g'(t)$ leads to

$$\overline{f\, g} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(\overline{f} + f'(t)\right)\left(\overline{g} + g'(t)\right) dt \tag{10}$$

or

$$\overline{f\, g} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(\overline{f}\, \overline{g} + \overline{g}\, f'(t) + \overline{f}\, g'(t) + f'(t)\, g'(t)\right) dt \tag{11}$$

or

$$\overline{f\, g} = \overline{f}\, \overline{g} + \lim_{T \to \infty} \frac{\overline{g}}{T} \int_{t_0}^{t_0+T} f'(t)\, dt + \lim_{T \to \infty} \frac{\overline{f}}{T} \int_{t_0}^{t_0+T} g'(t)\, dt + \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(f'(t)\, g'(t)\right) dt \quad . \tag{12}$$

According to Eq. (8) the second term and the third term on the right-hand side of this equation are equal to zero. Thus, we obtain

$$\overline{f\,g} = \bar{f}\,\bar{g} + \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(f'(t)\, g'(t) \right) dt \tag{13}$$

or

$$\overline{f\,g} = \bar{f}\,\bar{g} + \overline{f'\,g'} \tag{14}$$

with

$$\overline{f'\,g'} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(f'(t)\, g'(t) \right) dt \tag{15}$$

If $\overline{f'\,g'}$ could be expressed by $\overline{f'\,g'} = \overline{f'}\,\overline{g'}$ or

$$\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(f'(t)\, g'(t) \right) dt = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f'(t)\, dt \cdot \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} g'(t)\, dt \quad , \tag{16}$$

as suggested by Smith (2008) with his equation (8), we would have

$$\overline{f'\,g'} = 0 \quad . \tag{17}$$

Consequently, all variance or covariance terms that occur in the governing equations of turbulent systems would be equal to zero. This is sheer nonsense.